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Longitudinal and transverse forces on a vortex in superfluid ^{4}He

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Abstract. We examine the transverse and longitudinal components of the drag force upon a straight vortex line due to the scattering of liquid ^{4}He excitations. For this purpose, we consider a recently proposed Hamiltonian that describes the dissipative motion of a vortex, giving an explicit expression for the vortex–quasiparticle interaction. The dissipative coefficients involved are obtained in terms of the reservoir correlation function. Most of our explicit calculations are concerned with the range of temperatures below 0.4 K, for which the reservoir is composed of phonon quasiparticle excitations. We also discuss some important implications for the determination of possible scattering processes leading to dissipation, on the basis of the values of the vortex mass found in the literature.

Although it is well established that vortex dynamics plays an important role in the behaviour of superfluids, there are many controversial results in the literature concerning its dissipative motion. At zero temperature the motion of the vortex is provoked by the transverse Magnus force, but at finite temperatures the scattering of collective excitations by the vortex provides a dissipation mechanism that damps its cyclotron motion. It is widely accepted that for low enough temperatures, and only taking into account phonon–vortex scattering, the longitudinal component of this dissipative force behaves as the fifth power of the temperature [1]. Nevertheless, the form of the transverse component is still controversial [2], since very different results have been reported and it does not appear that any of them can be regarded as definitive.

In this context, we have recently presented [3] a model for the dissipation of a straight vortex line in superfluid ^{4}He in which the vortex is regarded as a macroscopic quantum particle whose irreversible dynamics can be described in the framework of the generalized master equations. This procedure enables us to cast the effect of the coupling between the vortex and the heat bath as a drag force with one reactive and one dissipative component, in agreement with phenomenological theories. In this paper we shall investigate the components of the drag force, considering the reservoir to be composed of the quasiparticle (qp) excitations of the superfluid. The dissipative vortex dynamics arises then from the scattering processes occurring between the vortex and the qps. Most of our explicit calculations will be concerned with the range of temperatures below 0.4 K, for which the qp excitations constitute a phonon reservoir [4].

Let us begin by giving a description of our model. Choosing a coordinate system fixed to the superfluid, the Hamiltonian for a straight vortex line parallel to the z-axis may be written as [3, 4]

$$H_v = \frac{1}{2m_v}\,[\boldsymbol{p} - q_v \boldsymbol{A}(\boldsymbol{r})]^2 \tag{1}$$

where $\boldsymbol{r}$ and $\boldsymbol{p}$ denote respectively the vortex position and momentum operators,

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{h\rho_s l}{2}(y, -x) \tag{2}$$

is the vector potential for the Magnus force, m_v is the inertial mass of the vortex, ρ_s the number density of the superfluid, h is Planck's constant, l is the system length along the z-axis and $q_v = \pm 1$ is the sign of the vorticity according to the right-handed convention.

In what follows we will consider the excitations to be at rest, so that the velocity of the normal fluid vanishes. In a previous work [5] we considered an interaction Hamiltonian of the form

$$H_{int} = -\boldsymbol{B} \cdot \boldsymbol{v} \tag{3}$$

where $\boldsymbol{B}$ and $\boldsymbol{v}$ are vectors that depend on the reservoir and vortex operators respectively. We have proven that the only linear combination of the vortex observables that leads to a dynamics in accordance with phenomenological descriptions is

$$\boldsymbol{v} = \left(\frac{p_x}{m_v} - \frac{\Omega}{2}y, \frac{p_y}{m_v} + \frac{\Omega}{2}x\right) \tag{4}$$

where Ω is the 'cyclotron frequency':

$$\Omega = \frac{q_v h\rho_s l}{m_v}. \tag{5}$$

In order to obtain an equation of motion for the mean value of the complex vortex position operator $R = x + \mathrm{i}y$, in previous works [3, 5] we derived, by means of a standard reduction–projection procedure, a generalized master equation for the density operator of the vortex. We employed the usual weak-coupling approximation, in which the vortex dynamics is affected by the reservoir degrees of freedom only through the second-order time correlation tensor $\langle \boldsymbol{B}(t)\boldsymbol{B}\rangle$, where the angular brackets indicate an average over the reservoir equilibrium ensemble and $\boldsymbol{B}(t)$ denotes a free time evolution for the reservoir operators. In addition, such a tensor is naturally assumed to be isotropic in the x–y plane. We have also made use of the Markovian approximation which assumes that such correlations are short lived within observational times. From such a master equation, we then extracted equations of motion for the expectation values of the vortex position and momentum operators. Finally, after some algebra and by elimination of the momentum, we obtained the desired equation for $\langle R(t)\rangle$ [3,5]:

$$m_v\langle \ddot{R}\rangle = \mathrm{i}(m_v\Omega + \gamma)\langle \dot{R}\rangle. \tag{6}$$

The complex coefficient γ is defined as

$$\gamma = \frac{2\Omega}{\hbar}\int_0^\infty \mathrm{d}\tau\ \psi(\tau)\mathrm{e}^{-\mathrm{i}\Omega\tau} \tag{7}$$

where $\psi(\tau)$ is the imaginary part of the time correlation isotropic tensor element:

$$\psi(\tau) = \Im(\langle B_j(\tau)B_j\rangle). \tag{8}$$

In the right-hand side of equation (6), one can identify two forces, namely the Magnus force [2] and the drag force. In particular, the drag force

$$F_D = \mathrm{i}\gamma\langle \dot{R}\rangle \tag{9}$$

has two components. One is parallel to the Magnus force, which we shall call the transverse force (TF):

$$F_T = \mathrm{i}\Re(\gamma)\langle\dot{R}\rangle \tag{10}$$

and the other one is parallel to the velocity and we shall refer to it as the longitudinal force (LF):

$$F_L = -\Im(\gamma)\langle\dot{R}\rangle. \tag{11}$$

We recall that it is well known that the LF must be opposite to the velocity (in our notation this means $\Im(\gamma) > 0$). With respect to the TF, great controversy still exists as regards its direction [1, 2, 6] and, even more so, its actual existence [7, 8].

In the present work, the reservoir vector operator $\boldsymbol{B}$ of equation (3) is chosen so as to describe qp scattering by the vortex. In terms of creation ($a_{\boldsymbol{q}}^+$) and annihilation ($a_{\boldsymbol{q}}$) operators for a qp of momentum $\boldsymbol{q}$, it reads

$$\boldsymbol{B} = \sum_{\boldsymbol{k},\boldsymbol{q}}(\boldsymbol{k}-\boldsymbol{q})\Lambda_{k,q}a_{\boldsymbol{k}}^+a_{\boldsymbol{q}} \tag{12}$$

where $\boldsymbol{k}-\boldsymbol{q}$ is the transferred momentum and $\Lambda_{k,q}$ a vortex–qp coupling constant which, in order to preserve isotropy, is assumed to depend only on the modulus of the qp momentum ($q = |\boldsymbol{q}|$).

To calculate the drag coefficients, we must first compute the correlation function in (8). Taking into account the time evolution $a_{\boldsymbol{q}}(t) = a_{\boldsymbol{q}}(0)\mathrm{e}^{-\mathrm{i}\omega_q t}$ and after some calculations, we have

$$\langle B_j(t)B_j\rangle = \frac{1}{2}\sum_{\boldsymbol{k},\boldsymbol{q}}|\Lambda_{k,q}|^2(k^2+q^2)\mathrm{e}^{\mathrm{i}(\omega_k-\omega_q)t}n(\omega_k)[1+n(\omega_q)] \tag{13}$$

where $n(\omega_q)$ denotes the mean number of qps of energy $\hbar\omega_q$:

$$n(\omega_q) = \langle a_{\boldsymbol{q}}^+a_{\boldsymbol{q}}\rangle = \frac{1}{\mathrm{e}^{\beta\hbar\omega_q}-1}. \tag{14}$$

Note that at temperature $T = 0$ the correlation function (13) vanishes due to the factor $n(\omega_k)$, yielding a vanishing drag force as expected. This behaviour would not be reproduced by a linear interaction in the qp operators, such as the one proposed in the Hamiltonian of reference [9].

The parameter γ (equation (7)) can be written in terms of the Fourier transform $\psi[\omega]$ of the imaginary part of the correlation function (13) as

$$\Re(\gamma) = \frac{4}{\mathrm{i}\hbar}\Omega\mathcal{P}\int_0^\infty \mathrm{d}\omega\,\frac{\omega}{\Omega^2-\omega^2}\psi[\omega] \tag{15}$$

$$\Im(\gamma) = \frac{2\pi}{\mathrm{i}\hbar}\Omega\psi[\Omega] \tag{16}$$

where $\mathcal{P}$ refers to the principal part and

$$\psi[\omega] = \frac{\mathrm{i}}{4}\sum_{\boldsymbol{k},\boldsymbol{q}}(k^2+q^2)|\Lambda_{k,q}|^2[n(\omega_q)-n(\omega_k)]\delta(\omega_k-\omega_q-\omega). \tag{17}$$

The above expressions (16) and (17) for the dissipative coefficient in (11) can be interpreted in terms of the scattering processes embodied in our interaction Hamiltonian. In fact, we first note that the vortex Hamiltonian (1) has an equally spaced level spectrum of separation $\hbar\Omega$ (the so-called Landau levels [10]) and, in addition, the operator $\boldsymbol{v}$, which couples the vortex to the reservoir qps, can be expressed, from (4), as a linear combination of creation and destruction operators of a vortex energy quantum [10]. Thus, according to (3) and (12), we may identify the scattering processes embodied in our model as those involving one vortex energy quantum $\hbar\Omega$ jointly with the creation and destruction of one qp. Such processes can also be identified, from (17), as follows. Each term proportional to $n(\omega_q)$ represents the process by which a qp of energy $\hbar\omega_q$ combines with a vortex quantum $\hbar\Omega$ to form a qp of energy $\hbar\omega_k = \hbar\omega_q + \hbar\Omega$ (this may be seen from (16) and the Dirac delta in (17)). The weight $n(\omega_q)$ of this process can be easily understood by taking into account the *thermal* origin of the 'incoming' qp of energy $\hbar\omega_q$ in contrast to the *interaction* origin of the 'outgoing' qp of energy $\hbar\omega_k$. Similarly, the terms weighted by $n(\omega_k)$ in (17) represent the annihilation of a qp of energy $\hbar\omega_k$ jointly with the creation of a vortex quantum $\hbar\Omega$ and a qp of energy $\hbar\omega_q$. Finally, we note that the positive sign of the dissipative coefficient $\Im(\gamma)$ arises from the factor $n(\omega_q) - n(\omega_k) > 0$ in (17), i.e. the weight of processes involving vortex energy loss must be greater than that of those causing vortex energy gain.

At low enough temperatures it is to be expected that only scattering processes involving phonons should be relevant. Therefore, we shall use the phonon dispersion relation $\omega_q = c_s q$ (where c_s is the sound velocity) and impose a cut-off momentum on the qps. This amounts to neglect of all of the scattering processes which do not yield a phonon as the 'outgoing' qp. Note that the 'incoming' *thermal* qp will certainly be a phonon. Under such an approximation, equation (17) can be written as

$$\psi[\omega] = \frac{\mathrm{i}}{4c_s^2}\int_0^\infty \mathrm{d}\omega'\, S(\omega', \omega'+\omega)[\omega'^2 + (\omega'+\omega)^2][n(\omega') - n(\omega'+\omega)] \tag{18}$$

where we introduce the so-called scattering function [11], defined as

$$S(\omega', \omega'') = \sum_{\boldsymbol{k},\boldsymbol{q}} |\Lambda_{k,q}|^2 \delta(\omega' - \omega_q)\delta(\omega'' - \omega_k) \tag{19}$$

which is related to the scattering of the environmental excitations between states of frequencies ω' and ω''.

The integral in (18) has to be solved numerically except in the limit $T \to 0$. In fact, for $\hbar\omega/k_B T \to \infty$ the gain term $n(\omega' + \omega)$ can be neglected and the loss one $n(\omega')$ 'filters out' all but the lowest-frequency 'incoming' phonons. This means that the factors accompanying $n(\omega')$ in (18) can be approximated to lowest order in ω'. In particular, the scattering function (19) is assumed to be a continuous symmetric function of both variables [11] satisfying $S(\omega', \omega) \simeq S(\omega)\omega'^p$ for $\omega' \to 0$. Thus equation (18) can be approximated for $T \to 0$ as follows:

$$\psi[\omega] \simeq \frac{\mathrm{i}}{4c_s^2}\omega^2 S(\omega)\int_0^\infty \mathrm{d}\omega'\, \omega'^p n(\omega') = \frac{\mathrm{i}}{4c_s^2} p!\zeta(p+1)\omega^2 S(\omega)\left(\frac{k_B T}{\hbar}\right)^{p+1} \tag{20}$$

where $\zeta(n)$ $(n \geqslant 2)$ denotes the Riemann zeta function.

It is expedient to notice that our model is unable to provide an *a priori* explicit form for the scattering function, because we treat vortex and qp excitations as separate entities which are assumed to interact by means of a generic Hamiltonian. Any additional information should be based upon experimental results or more fundamental theories. In fact, for the lowest-temperature domain, only theoretical results are at present available and they predict a T^5-dependence for the LF [1]. Hence, we set $p = 4$ in (20) and accordingly a low-frequency $\sim\omega^4$-behaviour for the scattering function. To perform numerical calculations which illustrate our results, we shall assume a simple form for the scattering function (19), as a generalization of the usual super-Ohmic dissipation with an exponential cut-off [12]:

$$S(\omega', \omega'') \propto \omega'^4 \mathrm{e}^{-\omega'/\omega_0}\omega''^4 \mathrm{e}^{-\omega''/\omega_0} \tag{21}$$

where ω_0 is a frequency cut-off parameter which allows us to select just the phonon part of the ^{4}He qp excitation spectrum, i.e. the frequencies below $\omega_c \simeq 1.2$ ps^{-1}. This is evident from figure 1 where we plot the one-variable scattering function $S(\omega, \omega)$ for $\omega_0 = 0.06$ ps^{-1}. In addition, we plot the frequency spectrum $f(\omega)$ of the normal-fluid density (i.e. $\rho_n = \int_0^\infty f(\omega)\, \mathrm{d}\omega$) for $T = 0.4$ K, which shows that at most the first half of the phonon spectrum makes a relevant contribution to the qp excitations by analogy with the similar behaviour for the one-variable scattering function.

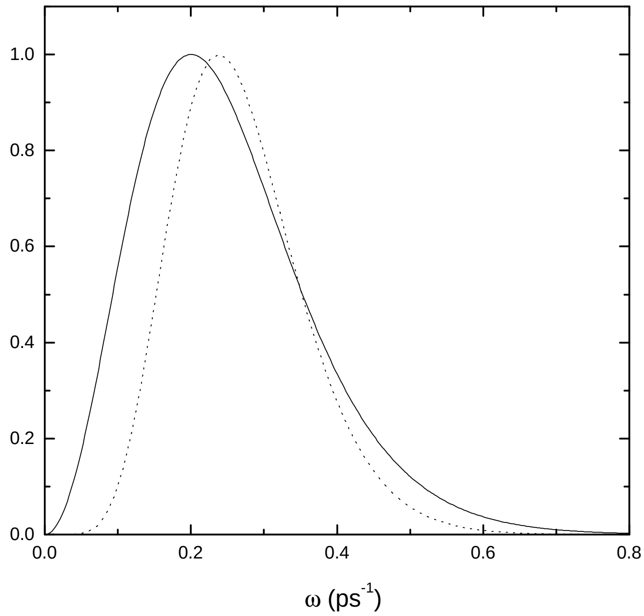

Figure 1. The one-variable scattering function, $S(\omega, \omega)$, for $\omega_0 = 0.06$ ps^{-1} (dotted line) and the frequency spectrum $f(\omega)$ of the normal-fluid density for $T = 0.4$ K (solid line). Different scales were used for the two functions in order to normalize the size of the two peaks.

From equations (15) and (16) we note that the value of the cyclotron frequency, which in turn depends on the vortex mass m_v (see equation (5)), is necessary for the determination of the drag force. Unfortunately, there is also controversy regarding the calculation of the vortex mass [9, 13–15]: possible values of Ω range from a lower bound [13] $\Omega_{min} \simeq 0.1$ ps^{-1} to the upper bound [4] $\Omega_{max} \simeq 3$ ps^{-1}. This suggests that a study of the dependence of the drag force on Ω could be informative.

In figure 2 we show the coefficients of both the transverse and the longitudinal components of the force—that is, $\Re(\gamma)$ and $\Im(\gamma)$ respectively—as functions of Ω for different temperatures. We see that, apart from a change of scale, the shape of the curves does not change appreciably across the temperature range $0 < T \leqslant 0.4$ K that we are considering. The only effect seems to be a shift to low frequencies with increasing temperatures (note that for a fixed vortex mass, the cyclotron frequency (5) decreases with increasing temperature due to the factor ρ_s; however, this variation is negligible for $T < 0.4$ K).

Regarding $\Im(\gamma)$ we first indicate that, according to (16), the plots of this function in figure 2 turn out to be proportional to the Fourier transform of $\dot{\psi}(t)$. In addition, we notice that $\Im(\gamma)$ vanishes for $\Omega > \omega_c$. Such an absence of dissipation is simply understood as a direct consequence of energy conservation in the scattering processes. This is imposed by the Dirac delta in (17), since it reflects the fact that phonon-to-phonon scattering events can only take place if the vortex energy quantum is not more energetic than the most energetic phonon of the

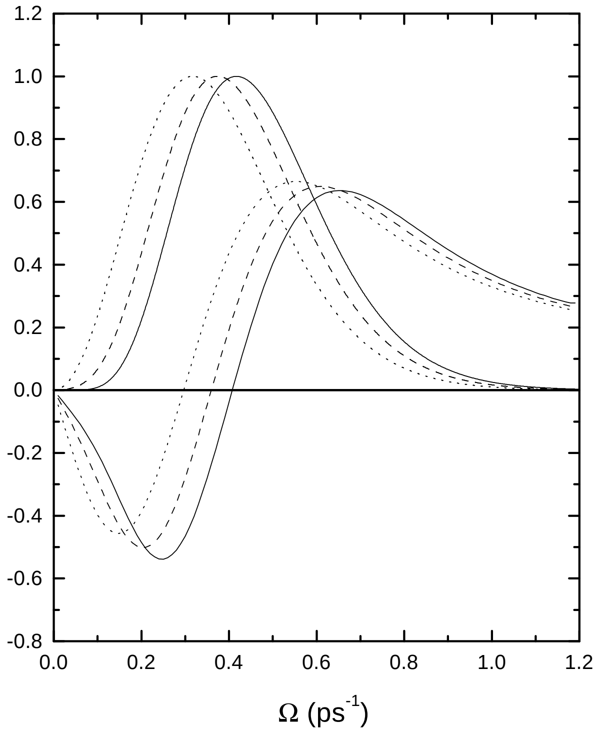

Figure 2. Coefficients of the TF ($\Re(\gamma)$) and the LF ($\Im(\gamma)$) for $T \to 0$ (equation (20)) (solid line), $T = 0.1$ K (dashed line) and $T = 0.4$ K (dotted line). Different scales were used for the different temperatures in order to normalize the size of the $\Im(\gamma)$ peak. The relative scale factors are 105.82 and infinity for $T = 0.4$ K$/T = 0.1$ K and $T = 0.1$ K$/T \to 0$, respectively. The exponential parameter $\omega_0 = 0.06$ ps^{-1} was used in equation (21).

spectrum. Therefore, we may conclude that dissipation due to phonon → phonon scattering should be negligible unless the vortex mass yields a value of the cyclotron frequency less than $\omega_c \simeq 1.2$ ps^{-1}, which is a value intermediate between the quoted frequencies Ω_{min} and Ω_{max}. Moreover, the value $\hbar\Omega_{max}$, arising from the usual hydrodynamical prescription for the calculation of the vortex mass [4], exceeds the energy of any undamped qp excitation [16]. So, we may extend our previous conclusion by saying that any dissipation via qp → qp scattering events should be negligible for such a value of the vortex mass, suggesting thus a possible scenario of dissipation via multi-qp scattering processes [17].

Regarding the TF, we remark that if it were possible to derive some experimental information about at least the direction of this component, this would also shed more light on the value of the vortex mass. In fact, from figure 2 we note that at values intermediate between Ω_{min} and ω_c, the TF changes its direction. In particular, a positive (negative) value of this component implies $\Omega > 0.45$ ps^{-1} ($\Omega < 0.3$ ps^{-1}), while a change of sign yields some intermediate value of Ω. Unfortunately, no experimental data are available for temperatures below 0.4 K.

It is important to note, in concluding this report, that our study has been restricted to a strictly rectilinear vortex, neglecting thus any possible contribution to the forces arising from vortex line curvatures. Classically, vortex lines can be deformed as helical waves known as Kelvin waves, and there is experimental evidence of similar modes for quantized vortices in helium II [4]. From the theoretical viewpoint, such waves could be regarded as a particular form of elementary excitation bound to the vortex. In fact, it has been shown that the elementary excitation spectrum of an imperfect Bose gas in the presence of a straight vortex line consists of both phonons and Kelvin-like waves [18]. Viewed from this perspective—two different kinds of independent elementary excitation—the consequence of scattering processes involving phonons and Kelvin modes appears eventually as a possible secondary effect upon the values of longitudinal and transverse forces. This issue has scarcely been addressed in the literature. In particular, Sonin long ago performed a simplified study [19] neglecting the vortex velocity due to Kelvin modes, but taking into account the effect of such modes on the scattering of phonons by means of an average over a classical Rayleigh–Jeans distribution for the oscillations of the vortex filament. He concluded that such an effect could at most modify the calculated forces for a strictly rectilinear vortex by 1.7% at $T = 0.5$ K. Unfortunately, within our formalism a quantitative study of the dynamics of a true three-dimensional curved vortex filament presents a high degree of difficulty.

In summary, starting from a microscopic model we have performed a calculation of the drag force on a moving vortex due to the scattering of qp excitations at temperatures below 0.4 K. We have also discussed, by analysing both the longitudinal and transverse forces as functions of the cyclotron frequency, some important implications in the determination of possible scattering processes leading to dissipation, on the basis of the values of the vortex mass found in the literature.

Acknowledgments

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[16] Excitations above two times the roton energy should be unstable against decay into two rotons. See
Pistolesi F 1998 *Phys. Rev. Lett.* **81** 397 and references therein
[17] Notice that such conclusions are based upon the Dirac delta energy conservation in (17) that derives from the Markov approximation. The lifetime for heat bath correlations $(\psi(t))$ can be estimated from the inverse of the frequency cut-off in (17) and it is determined by the highest undamped qp energy which is two times the roton energy. This yields a lifetime of order 0.4 ps which should be less than any observation time on the timescale of the dissipation $m_v/\Im(\gamma)$. Such a condition should be met for a sufficiently weak vortex–qp coupling, or equivalently for low enough temperatures. From Donnelly's book [4] we may identify $\Im(\gamma)/l = \gamma_0$, where the parameter γ_0 is a rapidly decreasing function for temperatures below 2 K. The lowest-temperature

measure of γ_0 (1.3 K) yields $m_v/\Im(\gamma) = q_v h \rho_s/(\Omega\gamma_0) = 26.8\Omega^{-1}$. Taking into account a value of Ω^{-1} of at least 1/3 ps, it is clear that the Markov approximation is very likely to be valid for temperatures below 1.3 K.

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